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# THE EQUILIBRIUM OF THE CONVERGENCE POINT IN TWO-STRAND YARN PLYING

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**Abstract**—In this paper the theory of the bending and twisting of thin rods of uniform circular cross-section is used to find the relationships between the ply-helix angle, the strand convergence angle, and the applied tensions and torques required to hold the plied strands illustrated in Fig. 1 in equilibrium.

The solution of this problem is of importance to the textile yarn manufacturing industry where singles yarns, made from long staple fibres, such as wool, are plied together in order to bind the surface fibres more effectively into the plied yarn structure. This produces warp yarns that are more abrasive resistant.

A formula for the relationship between the pre-twist in two initially straight strands, and the helix angle of the plied structure obtained when they are allowed to twist together into a balanced ply structure is also derived in Section 4. A balanced two-ply structure is one that will maintain its configuration without the application of external tension or torque. Balanced ply structures are also important in textile manufacturing processes. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

In this paper the theory of the bending and twisting of thin rods of uniform circular cross-section [Greenhill (1883), Love (1927) ch. 18] will be used to find the applied tensions and torques required to hold the plied strands illustrated in Fig. 1 in equilibrium.

The solution of this problem is of importance to the textile yarn manufacturing industry where singles yarns, made from long staple fibres, such as wool, are plied together in order to bind the surface fibres more effectively into the plied yarn structure. This produces warp yarns that are more abrasive resistant. The results of an investigation into the mechanics of yarn plying have been reported in a series of papers in the textile literature. This investigation was carried out in connection with the development of the *Sirospun* yarn manufacturing system in which a plied yarn is produced from two untwisted rovings in a single ring-spinning operation. The process is described in the first paper in the series by Plate and Lappage (1982). In the second paper, Emanuel and Plate (1982a) propose a theoretical model for the equilibrium of the convergence point, where the strands come together and enter the ply. In the third paper, Emanuel and Plate (1982b), describe an experimental model of yarn plying in which two rubber strands of uniform circular cross-section were twisted together. Their theoretical and experimental results are in very good agreement given the simplifying assumptions that necessarily underly their theory.

In the notation of this paper the over-all force and moment equilibrium equations for the stationary plied strands shown in Fig. 1 are respectively [cf. Emanuel and Plate (1982a) eqns (3) and (4)]:

$$T_0 = 2T_\infty \cos\alpha, \tag{1}$$

and

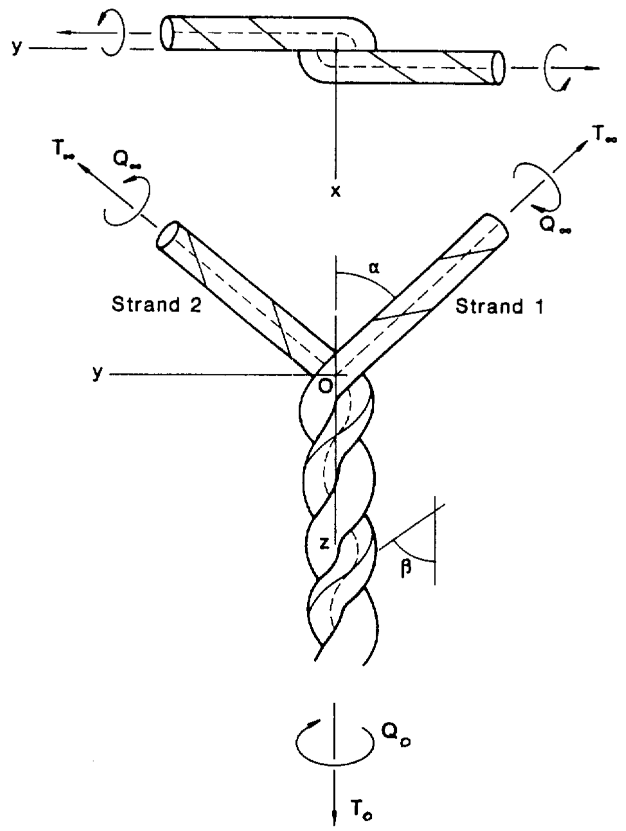


Fig. 1. The plied yarn structure, showing the cartesian axis system.

$$Q_0 = 2aT_\infty \sin\alpha + 2Q_\infty \cos\alpha, \tag{2}$$

where $a$ is the strand radius, $2\alpha$ is the convergence angle between the strands, $T_\infty$ is the tension, and $Q_\infty$ the torque in the strands above, and at some distance from the convergence point. $T_0$ is the tension, and $Q_0$ the torque applied to the plied yarn below the convergence point.

Miao *et al.* (1993) point out that these two equations are insufficient to determine the quantities $\alpha$, $Q_\infty$, and $T_\infty$ as functions of $T_0$, $Q_0$, and the ply-helix angle $\beta$. The theory of the Sirospun system developed in these papers does not depend on these overall equilibrium equations, but only on the assumption that the torsion in a strand as it lies in the ply is continuous with the torsion in the straight section of strand above the convergence point. It will be shown in this paper that this assumption is wrong, and that there is a small discontinuity in the strand torsion at the convergence point.

In this paper the theory of the bending and twisting of thin rods of uniform circular cross-section will be used to show that the convergence angle $\alpha$ is equal to the helix angle $\beta$, and since the helix angle can be determined from the known ply twist, this result, together with eqns (1) and (2) fully determines the three unknowns.

In an actual plying operation the strands are moving through the system with some constant feed speed $V$ and a full analysis of this system should strictly include the dynamical effects of this motion, and the rotational motion of the yarn. This will be taken up in a subsequent paper, Stump and Fraser (1997).

The theory of the bending and twisting of thin rods has a long history in the mathematical and the engineering literature. Much of this history and many references are given in the recent book by Antman (1995), and the papers of Thompson and Champneys (1996), and Champneys and Thompson (1996).

In the next section the derivation of the thin elastic rod equations as they apply to the present problem is reviewed, and in Section 3 their solution is given. The application of these results to yarn plying problems is discussed in Sections 4 and 5.

## 2. THE MATHEMATICAL FORMULATION

### 2.1. *The model assumptions*

The individual strands in the plied structure are modelled as initially straight elastic strands of uniform circular cross-section, radius $a$, and inextensible centre-line. The bending-moment/curvature and torque/torsion constitutive relations are assumed to be linear. In the textile yarn context torsion (radians/metre) is $2\pi$ times the twist (turns/metre). The equations for the bending and torsional stiffness respectively of a circular shaft are [Timoshenko and Young (1962)].

$$B = \tfrac{1}{4}EAa^2, \quad \text{and } K = \tfrac{1}{2}GAa^2, \tag{3}$$

where $E$ is the Young's modulus, $G$ is the shear modulus and $A = \pi a^2$ is the cross-sectional area of the shaft. If $\nu$ is Poisson's ratio then $G = E/[2(1+\nu)]$, and

$$K = \frac{B}{1+\nu}. \tag{4}$$

For an incompressible rubber strand $\nu = 0.5$, and $K = 2B/3$. The point to be made here is that these stiffnesses have the same order of magnitude.

Data given by Bennett and Postle (1979), and more recently by Tandon *et al.* (1995) show that singles yarns made from wool fibres have stiffnesses that are smaller than would be calculated by the formulae (3). However, the ratio of $K$ to $B$ given by eqn (4) is in approximate agreement with these data.

The axis of the strand in the ply is assumed to follow a helical path of radius $a$ and helix angle $\beta$, and the strands above the convergence point are assumed to be straight.

Finally it is necessary to make some assumptions about the forces of interaction between the strands as they lie in the ply, and at the convergence point. Clearly these forces must act in accord with the following two hypotheses:

- Since these are internal forces with respect to the system as a whole the force that strand 1 exerts on strand 2 must be equal and opposite to the force that strand 2 exerts on strand 1 (Newton's third law).
- If strand 1 is rotated through 180° about the $OZ$ axis (Fig. 1) it must coincide with the position of strand 2, and all the forces acting on the rotated strand 1 must coincide with the forces acting on strand 2.

There are two situations to be considered: the interaction between the strands as they lie in the ply, and the interaction between the strands just at the convergence point where it is necessary to allow for the possibility of concentrated point forces and moments of interaction.

- First, consider the interaction between the strands as they lie in the ply. Consistent with the above hypotheses, any friction or pressure between the strands in the ply must act on the surface of the strand and lie in a direction perpendicular to the ply axis ($OZ$ in Fig. 1). Any friction force resultant would therefore have components parallel and perpendicular to the strand axis which would give rise to variations in the magnitudes of the strand torque, bending moment and tension with distance along the strand. If it is assumed that these magnitudes are invariant with respect to translation along the ply axis then the force resultant of any surface shear stress distribution due to friction between the strands must be zero. Thus the only interaction force between the strands is a pressure p per unit strand acting in a direction normal to the axes of both strands.

- Second, at the convergence point the deformation of the strands will be highly three-dimensional and therefore only approximately represented by the one-dimensional rod theory. The pressure and frictional stress between the strands will vary rapidly in the neighbourhood of the convergence point over a distance that is less than the strand diameter.

  Only the resultant force and moment associated with this stress distribution can be taken into account in the context of the one-dimensional theory, and these resultants are represented by a point force and moment acting between the strands at the convergence point, in directions perpendicular to the ply axis (see eqns (7) and (9) below).

2.2. *Force and moment equilibrium equations*

Let $\mathbf{R}(s)$ be the position vector of the material point $P$ on the strand axis relative to a fixed origin $O$ on the ply axis at the strand convergence point (Fig. 1), and $s$ is distance measured along the strand axis. In the strand above the convergence point $s < 0$ and in the ply $s > 0$. Cartesian coordinates $Oxyz$ are defined in Fig. 1 with $Oz$ coinciding with the axis of the ply, and unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the usual cartesian basis vectors.

2.2.1. *Some geometrical preliminaries.* Because the strand axis is inextensible,

$$\mathbf{R}'\cdot\mathbf{R}' = 1, \tag{5}$$

where $()' = \mathrm{d}()/\mathrm{d}s$, and $\mathbf{R}'$ is the unit tangent vector to the strand axis.

The principal normal and binormal vectors, $\mathbf{n}$ and $\mathbf{b}$, respectively, for the helical strand path in the ply ($s > 0$) are defined as follows:

$$\mathbf{n} = \mathbf{R}''/|\mathbf{R}''|, \quad \text{and } \mathbf{b} = \mathbf{R}' \wedge \mathbf{n}. \tag{6}$$

Vectors $\mathbf{R}'$, $\mathbf{n}$ and $\mathbf{b}$ form a mutually orthogonal, right handed, triad of unit vectors. To ensure physical continuity of the strands through the convergence point $\mathbf{R}(s)$ and $\mathbf{R}'(s)$ must be continuous at $s = 0$, and in strand 1, for example, the identification $\mathbf{n} = -\mathbf{i}$ for $s < 0$ must be made.

Now consider the force and moment equilibrium of a right-cylindrical element of the strand of length $\delta s$ at $P$ (Fig. 2a).

2.2.2. *Force equilibrium.* Figure 2b shows the forces acting on the element: the shear force $\mathbf{V}$, the strand tension $T$ and the interaction force

$$\left.\begin{aligned} \mathbf{F} &= -p\mathbf{n}H(s)+\hat{\mathbf{F}}\delta(s),\\ \text{where}\quad \hat{\mathbf{F}} &= \hat{f}[-\mathbf{R}'\sin\beta+\mathbf{b}\cos\beta], \end{aligned}\right\} \tag{7}$$

$-p\mathbf{n}$ is the pressure per unit length of the strand axis exerted by strand 2 on strand 1 when the strands are lying in the ply, $\hat{f}$ and $\hat{m}$ are the magnitudes of the resultant point force and moment acting between the strands at the convergence point, $H(s)$ is Heaviside's unit step function, and $\delta(s)$ is Dirac's delta function.

Thus, the vector equation for the equilibrium of these forces is

$$(T\mathbf{R}')'+\mathbf{V}'-p\mathbf{n}H(s)+\hat{\mathbf{F}}\delta(s) = 0. \tag{8}$$

2.2.3. *Moment equilibrium.* Figure 2c shows the moments acting on element: the torque $Q$ acting in the tangential direction, and the bending moment $\mathbf{M}$ perpendicular to the tangential direction. To these must be added the moment $\mathbf{R}' \wedge \mathbf{V}$ (per unit length) due to the shear force, and the concentrated moment $\hat{\mathbf{M}}\delta(s)$ at the convergence point due to $\hat{f}$ and $\hat{m}$, where

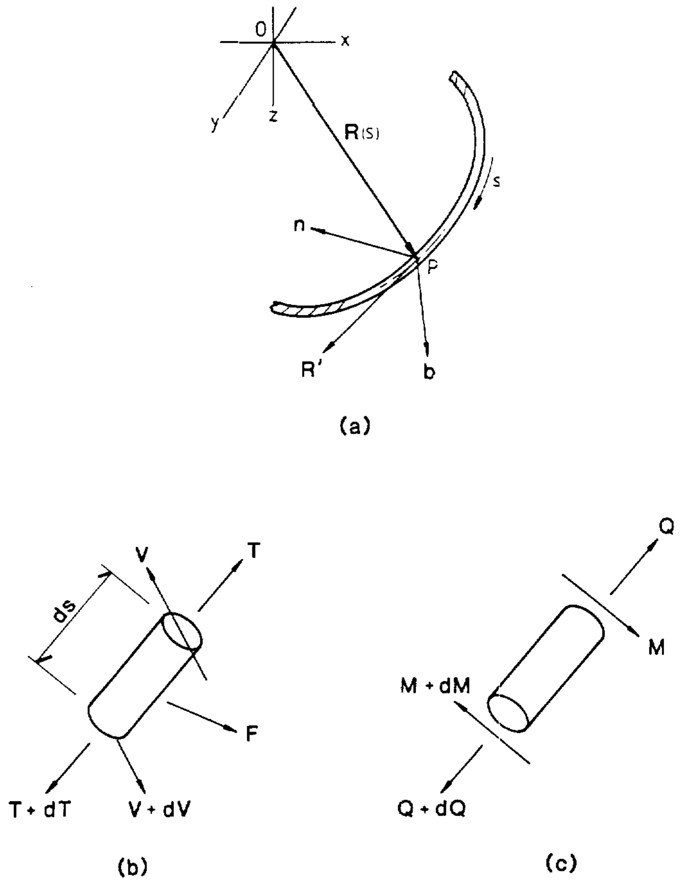


Fig. 2. Schematic diagrams showing the notation: (a) The position vector and unit tangent, principal normal and binormal vectors $\mathbf{R}'$, $\mathbf{n}$ and $\mathbf{b}$. (b) The forces acting on the yarn element, and (c) the moments acting on the yarn element.

$$\hat{\mathbf{M}} = a\hat{f}[\mathbf{R}'\cos\beta+\mathbf{b}\sin\beta]+\hat{m}[-\mathbf{R}'\sin\beta+\mathbf{b}\cos\beta]. \tag{9}$$

The moment equilibrium equation is

$$(Q\mathbf{R}')'+\mathbf{R}' \wedge \mathbf{V}+\mathbf{M}'+\hat{\mathbf{M}}\delta(s) = 0, \tag{10}$$

the bending moment/curvature relation

$$\mathbf{M} = B(\mathbf{R}' \wedge \mathbf{R}''), \tag{11}$$

and the constraint

$$\mathbf{V}\cdot\mathbf{R}' = 0, \tag{12}$$

complete the set of equations required to determine the path of the strand axis.

2.3. *Two first integrals*

Mielke and Holmes (1988) have shown that for general constitutive laws, including the case where axial extensibility is included, eqns (8) and (10) (with $\hat{\mathbf{F}} = \hat{\mathbf{M}} = 0$) admit two independent first integrals. The derivation for the present special case is now outlined.

To obtain the first result form the scalar product of the moment equilibrium equation (10) with the tangent vector $\mathbf{R}'$ to obtain the result $Q' = 0$. This means that the magnitude of the torque along the strand in the ply and above the convergence point is constant. This is true for any strand as long as the external force on the strand causes no moment component about the strand axis. However, if $\hat{\mathbf{M}} \neq 0$ there will be a discontinuity in $Q$ at the convergence point (at $s = 0$).

Note, that this result depends only on eqn (5), (11) and (10), and not on the linearity of the torque/torsion constitutive relation.

The derivation of the other first integral, which gives an explicit expression for the tension $T$, is a little more intricate. First form the scalar product of eqn (8) with the tangent vector $\mathbf{R}'$, and make use of eqn (5) to obtain

$$T' = -\mathbf{V}'\cdot\mathbf{R}' = \mathbf{V}\cdot\mathbf{R}''. \tag{13}$$

The last result on the right is obtained by differentiation of eqn (12).

An expression for the shear force $\mathbf{V}$ is found when the vector product of $\mathbf{R}'$ with eqn (10) is formed. After use is made of the expansion formula for the triple vector product, and account is taken of eqns (5) and (12), the result is

$$\mathbf{V} = Q(\mathbf{R}' \wedge \mathbf{R}'') + \mathbf{R}' \wedge \mathbf{M}'. \tag{14}$$

The moment/curvature relation (12) is now used to replace $\mathbf{M}'$ in this equation, and the final expression for $\mathbf{V}$ is substituted into eqn (13) to give the result:

$$\begin{aligned} T' &= \mathbf{R}''\cdot\{\mathbf{R}' \wedge B(\mathbf{R}' \wedge \mathbf{R}''')\} \\ &= -B\mathbf{R}''\cdot\mathbf{R}''' \\ &= -\tfrac{1}{2}B(\mathbf{R}''\cdot\mathbf{R}'')', \end{aligned}$$

where the expansion formula for the triple vector product, has again been used to obtain the second line in the above equation. This equation can now be integrated to give the final result:

$$T = T_g - \tfrac{1}{2}B\mathbf{R}''\cdot\mathbf{R}'', \tag{15}$$

where $T_g$, the constant of integration, is a reference tension (at a guide eye on the axis of rotation for example).

### 2.4. *Twist and torsion*

Love (1927) showed that the total torsion $\tau$ in an initially straight rod that is twisted and bent into a curved path (which is the case for the strand in the ply) is the sum of two parts:

1. The torsion $\mathrm{d}\phi/\mathrm{d}s$, where $\phi$ is the angle between the principal plane of curvature (defined by the vectors $\mathbf{R}'$ and $\mathbf{n}$) and a radial line, perpendicular to the strand axis joined to any straight line parallel to the axis marked on the surface of the initially straight untwisted strand. $\mathrm{d}\phi/\mathrm{d}s$ can also be interpreted as the initial torsion in the straight strand before its axis is deformed into a curved path.
2. The torsion due to the rotation of the principal plane of curvature, which is equal to $\pm|\mathbf{b}'|$. This quantity is called the *tortuosity* [Love (1927) ch. 18]. If the rotation of the principal plane of curvature is positive with respect to the intrinsic base vectors $\mathbf{R}'$, $\mathbf{n}$, $\mathbf{b}$ then the positive sign should be chosen.

From the third Frenet formula [cf. O'Neil (1991) p. 851] $\mathbf{b}' = -|\mathbf{b}'|\mathbf{n}$, and this result can be used to define a total torsion vector for a stand following a curved path as follows:

$$\tau\mathbf{R}' = \left(\frac{\mathrm{d}\phi}{\mathrm{d}s} - \mathbf{b}'\cdot\mathbf{n}\right)\mathbf{R}'. \tag{16}$$

For the ply structure illustrated in Fig. 1, since the strands are straight above the convergence point and follow a helical path (with helix angle $\beta$) below the convergence point, the torsion is given by

$$\tau = \begin{cases} \dfrac{d\phi}{ds}, & s < 0 \\ \dfrac{d\phi}{ds} + \dfrac{\sin\beta\cos\beta}{a}, & s > 0 \end{cases} \tag{17}$$

where $-\mathbf{b}'\cdot\mathbf{n} = \cos\beta\sin\beta/a$ is the tortuosity of the helical path (cf. eqn $(27)_3$ below), and neither $\tau$ nor $\phi'$ need be continuous across $s = 0$ in the above equation.

If the relationship between torque and torsion is linear, as assumed in this model, then

$$Q\mathbf{R}' = K\tau\mathbf{R}' = K\left(\frac{d\phi}{ds} - \mathbf{B}'\cdot\mathbf{n}\right)\mathbf{R}', \tag{18}$$

where $K$ is the torsional stiffness defined in eqn (3).

The relationship between twist and torsion in plied yarn structures has been discussed in detail by Treloar (1956). In the present paper 'Z' twist, which corresponds to the positive tortuosity of the right-handed helix followed by a strand in the ply illustrated in Fig. 1, is positive. In the textile yarn context the strand twist (turns per unit length) is defined by

$$T_W = \frac{1}{2\pi}\frac{d\phi}{ds}. \tag{19}$$

In order to obtain the present solution, in the context of thin elastic rod theory, it is necessary to introduce concentrated moment and force resultants to describe the interaction between the strands at the convergence point. This results in a discontinuity in the strand torque, and hence the torsion, at the convergence point. The significance of this for understanding textile yarn plying operations is discussed in Section 5.

This completes the mathematical formulation of the equations that determine the path of a plied strand. They will now be recast in terms of dimensionless variables appropriate to the problem under consideration.

### 2.5. *The dimensionless equations*

Lengths are scaled against the yarn radius $a$, and forces are scaled against $B/a^2$, which is a measure of the magnitude of the forces required to bend and twist the strands. Moments are scaled against $B/a$.

$$\begin{gathered} \bar{\mathbf{R}} = \frac{\mathbf{R}}{a} = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}, \quad \bar{s} = \frac{s}{a}, \\ \bar{\mathbf{V}} = \frac{\mathbf{V}a^2}{B}, \quad \bar{T} = \frac{Ta^2}{B}, \quad \bar{p} = \frac{pa^3}{B}, \quad \hat{\mathbf{F}} = \frac{\hat{\mathbf{F}}a^2}{B}, \\ \bar{Q} = \frac{Qa}{B}, \quad \bar{\mathbf{M}} = \frac{\mathbf{M}a}{B}, \quad \hat{\mathbf{M}} = \frac{\hat{\mathbf{M}}a}{B}, \\ \bar{B} = 1, \quad \kappa = \frac{K}{B}, \quad \bar{T}_W = T_W a. \end{gathered} \tag{20}$$

In terms of these dimensionless variables the force equilibrium eqn (8) becomes

$$(\bar{T}\bar{\mathbf{R}}')' + \bar{\mathbf{V}}' - \bar{p}\mathbf{n}H(\bar{s}) + \hat{\bar{\mathbf{F}}}\delta(\bar{s}) = 0, \tag{21}$$

where

$$\hat{\bar{\mathbf{F}}} = \hat{\bar{f}}[-\bar{\mathbf{R}}'\sin\beta + \mathbf{b}\cos\beta], \tag{22}$$

and the moment eqn (10) becomes

$$(\bar{Q}\bar{\mathbf{R}}')' + \bar{\mathbf{R}}' \wedge \bar{\mathbf{V}} + \bar{\mathbf{M}}' + \hat{\bar{\mathbf{M}}}\delta(\bar{s}) = 0, \tag{23}$$

where

$$\hat{\bar{\mathbf{M}}} = \hat{\bar{f}}[\bar{\mathbf{R}}'\cos\beta + \mathbf{b}\sin\beta] + \hat{\bar{m}}[-\bar{\mathbf{R}}'\sin\beta + \mathbf{b}\cos\beta]. \tag{24}$$

Equations (5), (11) and (12) become

$$\bar{\mathbf{R}}'\cdot\bar{\mathbf{R}}' = 1, \quad \bar{\mathbf{M}} = \bar{\mathbf{R}}' \wedge \bar{\mathbf{R}}'', \quad \bar{Q} = \kappa\tau, \quad \text{and} \quad \bar{\mathbf{R}}\cdot\bar{\mathbf{V}} = 0. \tag{25}$$

## 3. SOLUTION OF THE TITLE PROBLEM

As is well known the nonlinear equations for a thin elastic rod of uniform circular cross section have two exact solutions:

1. The strand, subject to a constant tension $T$ and torsion $Q = K(\mathrm{d}\phi/\mathrm{d}s)$, remains straight. As Greenhill showed this solution is stable for an infinitely long strand provided that $T > Q^2/4B$. [See also Thompson and Champneys (1996) eqn (2.1).] The application of this stability criteria in the textile yarn context has been discussed in the book by Hearle *et al.* (1969) (section 1.7). This solution is appropriate for the segment of the strand above the convergence point ($s < 0$).
2. Under appropriate application of forces and moments at its ends, the strand axis deforms into a helix. This solution is appropriate for the strand as it lies in the ply ($s > 0$).

In the remainder of the paper the bar over the dimensionless variables will be dropped, as all variables are dimensionless unless otherwise stated.

In order to join these two solutions together smoothly at the convergence point $s = 0$ it is necessary to introduce the concentrated force and moment resultants defined in eqns (7) and (9). Since the tangent vector $\mathbf{R}'$ must be continuous as the strand passes into the ply the convergence half-angle must be equal to the helix angle: $\alpha = \beta$, so that the position vector of the path for strand 1 (Fig. 1) is given by

$$\mathbf{R}(s) = \begin{Bmatrix} \mathbf{i} + \mathbf{j}s\sin\beta + \mathbf{k}s\cos\beta, & s < 0, \\ \mathbf{i}\cos\theta + \mathbf{j}\sin\theta + \mathbf{k}s\cos\beta, & s > 0, \end{Bmatrix} \tag{26}$$

where ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) are the unit base vectors of the cartesian coordinate system (Fig. 1), and $\theta' = \sin\beta$, $s > 0$. In strand 1, $\theta(0) = 0$, and in strand 2, $\theta(0) = \pi$.

### 3.1. *Solution for the strand in the ply*

In the ply $s > 0$, and the strand axis follows the right-handed helical path given by eqn $(26)_2$. The tangent, principal normal, and binormal unit vectors for this path are, respectively,

$$\begin{aligned} \mathbf{R}' &= (-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)\sin\beta + \mathbf{k}\cos\beta, \\ \mathbf{n} &= -\mathbf{i}\cos\theta - \mathbf{j}\sin\theta, \\ \mathbf{b} &= (\mathbf{i}\sin\theta - \mathbf{j}\cos\theta)\cos\beta + \mathbf{k}\sin\beta. \end{aligned} \tag{27}$$

From eqns $(17)_2$ and (18), $(25)_2$ and (14), respectively, the torque, bending moment and shear force in the strand are found to be

$$Q = \kappa\left\{\frac{d\phi}{ds} + \cos\beta\sin\beta\right\},$$
$$\mathbf{M} = \sin^2\beta\mathbf{b},$$
$$\mathbf{V} = \{Q\sin^2\beta - \cos\beta\sin^3\beta\}\mathbf{b}. \tag{28}$$

The (dimensionless) tension and torque applied to the ply are expressed in terms of these strand variables and the strand tension $T$ as follows:

$$\begin{aligned} T_0\mathbf{k} &= T(\mathbf{R}_1' + \mathbf{R}_2') + \mathbf{V}_1 + \mathbf{V}_2 \\ &= \{2T\cos\beta + 2Q\sin^3\beta - 2\cos\beta\sin^4\beta\}\mathbf{k}, \\ Q_0\mathbf{k} &= \mathbf{M}_1 + \mathbf{M}_2 + Q(\mathbf{R}_1' + \mathbf{R}_2') \\ &\quad + T(\mathbf{R}_1 \wedge \mathbf{R}_1' + \mathbf{R}_2 \wedge \mathbf{R}_2') + \mathbf{R}_1 \wedge \mathbf{V}_1 + \mathbf{R}_2 \wedge \mathbf{V}_2 \\ &= \{2T\sin\beta + 2Q\cos^3\beta + 2\sin^3\beta(1+\cos^2\beta)\}\mathbf{k}, \end{aligned} \tag{29}$$

where subscripts 1 and 2 indicate that the variable is to be evaluated for strand 1 and strand 2, respectively, and the expressions for **V** and **M** given in eqns (28) have been used to obtain the second lines in the above results for $T_0$ and $Q_0$.

When these two equations are solved for the strand tension $T$ and torque $Q$. The result is

$$T = \frac{1}{2\cos 2\beta}(-Q_0\sin^3\beta + T_0\cos^3\beta + 2\sin^4\beta),$$
$$Q = \frac{1}{2\cos 2\beta}(Q_0\cos\beta - T_0\sin\beta - 4\cos\beta\sin^3\beta). \tag{30}$$

To find an expression for the pressure $p$ between the strands form the scalar product of eqn (21) with the principal normal **n** and make use of results $(27)_2$, and (30) to obtain

$$\begin{aligned} p &= T\mathbf{R}''\cdot\mathbf{n} + \mathbf{V}'\cdot\mathbf{n} \\ &= \frac{\sin^2\beta}{2\cos 2\beta}(-Q_0\sin\beta + T_0\cos\beta + 2\sin^2\beta). \end{aligned} \tag{31}$$

3.2. *Solution above the convergence point*

The strand axis is assumed to be straight for $s < 0$, the strand path is given by eqn $(26)_1$ and the tangent vector is

$$\mathbf{R}' = \mathbf{j}\sin\beta + \mathbf{k}\cos\beta, s < 0. \tag{32}$$

For $s < 0$ the strands lie in planes parallel to the $Oyz$ plane and a dimensional distance $2a$ apart.

Since $\mathbf{R}'$ is constant

$$\mathbf{V} = \mathbf{M} = 0, \quad Q = Q_\infty, \quad \text{and} \quad T = T_\infty. \tag{33}$$

This solution satisfies eqns (21) and (23) identically in the region $s < 0$. Finally, the dimensionless form of the overall equilibrium equations (1) and (2) with $\alpha = \beta$ can now be solved to give

$$T_\infty = \frac{T_0}{2\cos\beta},$$

$$Q_\infty = \frac{Q_0 - T_0\tan\beta}{2\cos\beta}. \tag{34}$$

The straight strand stability condition [Greenhill (1883)], which is $T_\infty > Q_\infty^2/4$ (in dimensionless terms) for this solution, must also be satisfied. This puts a limitation on the magnitude of the torque that can be applied at the bottom of the ply without causing the straight strands above the convergence point to buckle:

$$Q_0 < T_0\tan\beta + \sqrt{8T_0\cos\beta}. \tag{35}$$

3.3. *Joining the solutions at the convergence point*

Finally, it is necessary to show that the straight strand solution above the convergence point can be joined smoothly onto the helical strand path solution in the ply. To complete the solution, formulae for the magnitudes of the concentrated moment $\hat{m}$ and friction force $\hat{f}$ at the convergence point must be derived.

First note that the variables **V**, **M**, $T$, and $Q$ must all be discontinuous at $s = 0$, and this discontinuity is used to determine $\hat{f}$ and $\hat{m}$ as follows:

Integrate the force and moment equilibrium eqns (21) and (23) across the origin from $s = 0^-$ to $s = 0^+$, to obtain the jump conditions

$$\begin{aligned}
[T] &= \frac{\sin\beta}{2\cos 2\beta}\left(-Q_0\sin^2\beta + T_0\frac{\sin^3\beta}{\cos\beta} + 2\sin^3\beta\right) = \hat{f}\sin\beta,\\
[\mathbf{V}] &= \frac{\cos\beta}{2\cos 2\beta}\left(Q_0\sin^2\beta - T_0\frac{\sin^3\beta}{\cos\beta} - 2\sin^3\beta\right)\mathbf{b} = -\hat{f}\cos\beta\mathbf{b},\\
[Q] &= \frac{1}{2\cos\beta\cos 2\beta}\left(Q_0\sin^2\beta + T_0\frac{\sin^3\beta}{\cos\beta} - 4\cos^2\beta\sin^3\beta\right)\\
&= \hat{m}\sin\beta - \hat{f}\cos\beta,\\
[\mathbf{M}] &= \sin^2\beta\mathbf{b} = [-\hat{m}\cos\beta - \hat{f}\sin\beta]\mathbf{b},
\end{aligned} \tag{36}$$

where on the left hand sides $[T] = T(0^+) - T(0^-)$ and so on. Thus, there are four equations for the two unknowns $\hat{f}$ and $\hat{m}$. However this redundancy is only apparent as only two of these equations are independent. The solution to the system is

$$\begin{aligned}
\hat{f} &= \frac{1}{2\cos 2\beta}\left(-Q_0\sin^2\beta + T_0\frac{\sin^3\beta}{\cos\beta} + 2\sin^3\beta\right),\\
\hat{m} &= \frac{\sin\beta}{2\cos 2\beta}\left(Q_0\frac{\sin^2\beta}{\cos\beta} + T_0\frac{\sin^3\beta}{\cos^2\beta} - 2\cos 2\beta\sin\beta\right).
\end{aligned} \tag{37}$$

Finally, the value of $\phi'$ in the strands can be calculated from eqns (17), (18), (19), $(30)_2$ and $(34)_2$. The result is:

$$\phi' = \begin{cases} \dfrac{Q_0 - T_0 \tan\beta}{2\kappa\cos 2\beta} & s < 0, \\ \dfrac{1}{2\kappa\cos 2\beta}(Q_0\cos\beta - T_0\sin\beta - 4\cos\beta\sin^3\beta) - \cos\beta\sin\beta, & s > 0. \end{cases} \quad (38)$$

This completes the solution of the title problem. All the unknown quantities $T$, $T_\infty$, $Q$, $Q_\infty$, $\mathbf{M}$, $\mathbf{V}$, $p$, $\hat{f}$, $\hat{m}$, and $\phi'$ have now been expressed as functions of the applied torque and tension $Q_0$, $T_0$, the ply helix angle $\beta$, and the ratio of the torsional and bending stiffnesses $\kappa = K/B = 1/(1+\nu)$, where $\nu$ is Poisson's ratio. In the next two sections the application of this solution to two problems of interest to textile yarn manufacturers is discussed.

## 4. THE BALANCED PLY STRUCTURE

Balanced ply structures in which two strands of yarn are pretwisted and then plied together to form a structure that requires no applied torque or tension to maintain its equilibrium configuration are important in the textile yarn industry.

To create such a plied yarn structure the 'singles' strands are first both given (say) a left handed twist ($S$-twist) and then they are plied together in the opposite right handed ($Z$-twist) direction.

A balanced ply structure can be created by twisting a long single strand with pre-twist $-\phi'_0$ in the left hand ($S$-twist) direction and then doubling it length-wise upon its self and releasing the folded end of the strand so that the two strands 'snarl' into a plied $Z$-twist helix structure with helix angle $\beta$. The precise behaviour near the ends of the strands is complicated, but away from the ends the equations for the quantities in the strand are obtained from the results of Section 3.1 with $T_0 = Q_0 = 0$,

From eqns (30), $(28)_3$, and (31) the tension, torque, and shear force in the strands, and the pressure between the strands are found to be

$$T = \frac{\sin^4\beta}{\cos 2\beta},$$

$$Q = -\frac{2\cos\beta\sin^3\beta}{\cos 2\beta},$$

$$\mathbf{V} = -\frac{\cos\beta\sin^3\beta}{\cos 2\beta}\mathbf{b},$$

$$p = \frac{\sin^4\beta}{\cos 2\beta}. \quad (39)$$

The relation between the residual twist in the plied strand and the ply helix angle is obtained from eqn $(38)_2$.

$$\phi' = -\frac{2\cos\beta\sin^3\beta}{\kappa\cos 2\beta} - \cos\beta\sin\beta. \quad (40)$$

In order to find a relationship between the pretwist $-\phi'_0$ and the ply helix angle an additional equation is required. Since the strand bending and torsional behaviour in this model is linear elastic, and since there is no friction between the strands, and the initial and final states of the system before and after plying are in static equilibrium the pretwist elastic energy (per unit strand length) must be equal to the elastic energy stored in the final structure. That is

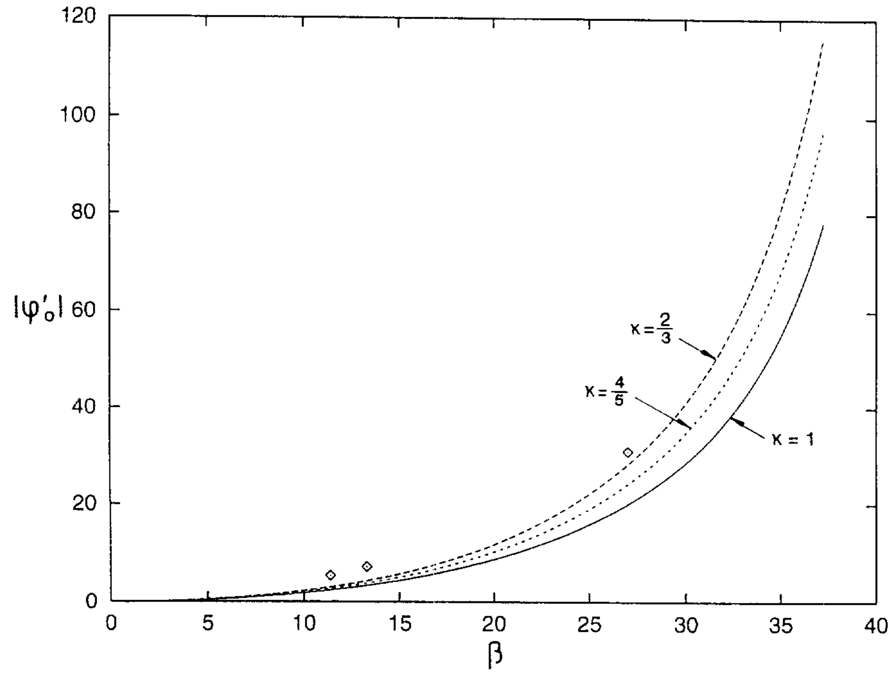


Fig. 3. Plot of eqn (42) showing the relationship of the pretwist $|\phi_0'|$ helix angle $\beta$ for a *balanced* ply. The plotted points were obtained experimentally.

$$\frac{1}{2}\kappa(\phi_0')^2 = \frac{1}{2}\frac{Q^2}{\kappa} + \frac{1}{2}|\mathbf{M}|^2, \tag{41}$$

and when eqns $(28)_2$ and $(39)_2$ are used to eliminate $\mathbf{M}$ and $Q$ from this equation the result is

$$\phi_0' = -\frac{\sin^2\beta}{\sqrt{\kappa}}\left(\frac{\tan^2 2\beta}{\kappa}+1\right)^{1/2}. \tag{42}$$

For values of $\nu = 0-1/2$, $\kappa = 1-2/3$. Figure 3 shows a plot of $|\phi_0'|$versus the helix angle $\beta$ for the balanced ply structure. Also shown in Fig. 3 are three experimental data points obtained from measurements of the writhing of thin-wall hard plastic tubing of 2 and 3 mm radius. These rough measurements made with only modest equipment confirm the trends of Fig. 3. More detailed experimental work is necessary for confirmation of the exact values of Fig. 3, but the gross trends are certainly observed.

## 5. APPLICATION TO THE THEORY OF THE SIROSPUN PROCESS

Dimensionless quantities are barred in this section as the final results given below are expressed in dimensional quantities.

As part of their experimental program in connection with the development of the Sirospun process Emanuel and Plate (1982a) derive a formula for the relationship between the strand twist $T_s$ (turns per unit strand length) above the convergence point and the ply twist $T_p$ (turns per unit length of the ply axis). Their result is [Emanuel and Plate (1982a) eqn (6)]

$$T_s = \frac{T_p}{1+4\pi^2a^2T_p^2}. \tag{43}$$

This result is derived on the assumption that the strand torsion is continuous at the convergence point, and that in the ply the strand torsion is due only to the tortuosity of the helical path of the strand axis. That is $\phi' = 0$ in eqn $(38)_2$. In the thin elastic rod theory solution derived here the torsion is discontinuous at the convergence point and formula (43) above must be amended as follows.

With the assumptions that $\phi' = 0$, $s > 0$ in the present theory, all quantities can be expressed in terms of $\bar{T}_0$, $\beta$, and $\kappa$. In particular, eqn $(38)_2$ gives

$$\bar{Q}_0 = \bar{T}_0 \tan\beta + 4\sin^3\beta + 2\kappa\sin\beta\cos 2\beta, \tag{44}$$

and when this result is substituted into eqn $(34)_2$ the result is

$$\hat{Q}_\infty = \kappa\cos\beta\sin\beta + (2-\kappa)\frac{\sin^3\beta}{\cos\beta}. \tag{45}$$

Thus the twist in the strand above the convergence point is

$$T_s = \frac{\bar{Q}_\infty}{2\pi\kappa a} = \frac{1}{2\pi a}\left[\cos\beta\sin\beta + \frac{(2-\kappa)}{\kappa}\frac{\sin^3\beta}{\cos\beta}\right]. \tag{46}$$

In terms of the yarn radius and the ply-helix angle $T_p$ is given by the equation

$$T_p = \frac{\tan\beta}{2\pi a}, \tag{47}$$

so that

$$\sin\beta = \frac{2\pi a T_p}{\sqrt{1+4\pi^2 a^2 T_p^2}}, \quad \cos\beta = \frac{1}{\sqrt{1+4\pi^2 a^2 T_p^2}}. \tag{48}$$

When these results are used in eqn (46) the final result is

$$T_s = \frac{T_p}{1+4\pi^2 a^2 T_p^2}\left[1 + \frac{(2-\kappa)}{\kappa}4\pi^2 a^2 T_p^2\right]. \tag{49}$$

Equations (43) and (49) differ by a term which is typically less than one tenth of the result (43), so that for moderate ply twist (small helix angles $\beta$) the correction from a practical point of view is negligible. Emanuel and Plate (1982b) were able to verify eqn (43) for ply twists up to $T_0 = 100$ turns per metre in their experiments with rubber strands, and up to $t_0 = 200$ turns per metre using a multifilament yarn.

## 6. CONCLUDING REMARKS

In this paper it has been shown how the bending moment, torque and tension in strands plied together as illustrated in Fig. 1 can be determined from the theory of twisted and bent rods by patching together the exact helical centreline solution of these equations for the strand in the ply with the exact straight centre-line solution for the strand above the convergence point. This is made possible by the introduction of concentrated force and moment interactions between the strands at the convergence point.

This problem was motivated by our wish to improve existing theories of two-ply yarn manufacturing processes. In fact it is the first step on the way to developing a comprehensive theory of yarn plying. In the next step we shall include the dynamical terms for the moving threadline in this model (Stump and Fraser 1997). The final object of the project is to create a theory of twist and ply insertion by the Sirospun process that will enhance our understanding of this process and its further developments currently being explored by the textile industry.

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